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DIFFUSION LIMIT FOR MANY PARTICLES IN A PERIODIC STOCHASTIC ACCELERATION FIELD

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The one-dimensional motion of any number $\mathcal{N}$ of particles in the field of many independent waves (with strong spatial correlation) is formulated as a second-order system of stochastic differential equations, driven by two Wiener processes. In the limit of vanishing particle mass $\mathfrak{m} \to 0$, or equivalently of large noise intensity, we show that the momenta of all $\mathcal{N}$ particles converge weakly to $\mathcal{N}$ independent Brownian motions, and this convergence holds even if the noise is periodic. This justifies the usual application of the diffusion equation to a family of particles in a unique stochastic force field. The proof rests on the ergodic properties of the relative velocity of two particles in the scaling limit.

1. Introduction. The motion of a particle in the field of many waves [DoGu03, DoMa06, Ts91] is a fundamental process in classical physics, the understanding of which is a prerequisite to the analysis of many plasma and fluid phenomena. In one space dimension, it can be described by the Hamiltonian model

$$(1.1) \qquad H = \frac{p^2}{2\mathfrak{m}} + \sum_{m=1}^{\mathcal{M}} A_m \cos(k_m q - \omega_m t - \varphi_m)$$

where the particle with mass $\mathfrak{m}$ has position q and momentum p, while the force field derives from a potential with time Fourier components $A_m \mathrm{e}^{\mathrm{i}\varphi_m}$. The wave field comprises $\mathcal{M}$ waves, with a smooth dispersion relation associating a wavenumber k_m, a pulsation ω_m and a phase velocity $v_m = \omega_m/k_m$ to each wave – usually determined by fixed properties of the environment, such as the geometry of the domain where waves propagate (then wavenumbers k_m and pulsations ω_m are discrete). The complex amplitudes $A_m \mathrm{e}^{\mathrm{i}\varphi_m}$

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are more easily tuned by the experimenter or affected by simple changes in the environment.

The dynamical systems approach to this problem discusses the particle motion after prescribing a single choice for each wave complex amplitude. As it would be quite exceptional to control all waves (though this is e.g. the assumption underlying the standard map, see [BeEs98b] for a discussion), physicists often turn to a probabilistic description of the dynamics, considering an "ensemble" of realizations (A_m, φ_m). Various arguments are then invoked to reduce the particle evolution equations to a stochastic differential equation, often driven by a "white noise". This results in somewhat tractable models (see e.g. [Kr02] about the validity of such derivations).

In this paper we focus on two issues. First, a random field characterized by (A_m, φ_m) for a given dispersion relation with discrete frequency spectrum may be periodic in time : may the force on the particle be considered as independent over several time periods in a genuine limit ? Second, may one consider several particles subject to the same wavefield as independent in a genuine limit ? The latter issue underlies the frequent application of the Fokker–Planck equation to the evolution of a family of particles in a single turbulent wavefield – though a priori one can only grant that the diffusion equation describes the evolution of the distribution of a single particle for an ensemble of wavefield samples.

From a more general perspective, this work also relates to the issue of "propagation of chaos" in statistical physics [Ka56, Ka59], an aspect of Hilbert's 6th problem : how does chaotic dynamics enable a system, in which initial data are independent ("random") but the evolution may generate correlations, to behave as if the evolution regenerated independence ("randomness") or destroyed correlations ? Here, how do two Wiener processes, fully describing a prescribed "turbulent" environment, generate $\mathcal{N}$ independent Brownian motions for particles ?

A further motivation for the present work is that physics literature most often focuses on the evolution of particle distribution functions, e.g. by showing that they obey a Fokker–Planck equation, and on instantaneous observables such as $p(t_1)$ for given t_1 (pointwise in time). However, the notion of a diffusion process implies rather a measure on the set of trajectories, viz. functions $p(\cdot)$ (globally in time). Here we shall show how our model implies that an arbitrary number of trajectories in a single realization of the dynamics do, jointly, admit the Wiener measure description.

In sections 2 and 3 we motivate the mathematical model more precisely and state our main results, which are proved in the subsequent sections. The crucial Theorem 3.2 is an ergodic theorem for a rescaled process, im-

plying that a process V_t, describing the relative velocity of one particle with respect to another one (or to its own earlier motion), converges weakly to a Brownian motion. The difficulty in proving the ergodic theorem is that the invariant measure of the diffusion process is infinite (it is the Lebesgue measure $\mathrm{d}x\mathrm{d}y$), and we must estimate a continuous additive functional generated by a function (namely $f(x,y) = \sin^2 x$) which is not integrable with respect to this measure (it is only locally integrable). This weak convergence then implies the final many–particle result (Theorem 3.1) by a straightforward application of the Lévy characterization of Brownian motion. Section 8 outlines implications and possible extensions to this work.

2. Physical background. Because the waves have different frequencies and velocities, it is generally unrealistic to assume their phases to be correlated. Their intensities are more easily observed, but both in nature and in the laboratory the accumulation of statistical data on waves often involves only their average power spectra, not the detailed intensity data for each measurement run. We assume here that these complex amplitudes are random data, and investigate the statistics of the particle motion in the resulting time–dependent random field. This dynamics is a "stochastic acceleration problem" for a "passive particle" in weak plasma turbulence [DiKr86, DoGr82, MaEl82, St66, VE97], and its understanding is a prerequisite to a proper analysis of the case where the particle motion feeds back on the wave evolution [DoCa97, ElEs03].

The Hamiltonian (1.1) generates equations of motion

$$(2.1) \qquad \dot{q} = p/\mathfrak{m},$$
$$(2.2) \qquad \dot{p} = \sum_m k_m A_m \sin(k_m q - \omega_m t - \varphi_m).$$

An important observation [Chi79, Es85] on the motion of a particle in the field (2.2) is locality in velocity : the evolution of the particle when it has velocity $\dot{q} = v$ depends only weakly on the waves with a Doppler-shifted frequency $\omega_m - k_m v$ much larger than their trapping oscillation frequency $k_m \sqrt{A_m/\mathfrak{m}}$. In particular, for a two-wave system the resonance overlap parameter

$$s_{1,2} = \frac{2\sqrt{A_1/\mathfrak{m}} + 2\sqrt{A_2/\mathfrak{m}}}{|\omega_2/k_2 - \omega_1/k_1|}$$

becomes unity when there exists a velocity $u = \omega_2/k_2 - 2\sqrt{A_2/\mathfrak{m}} = \omega_1/k_1 + 2\sqrt{A_1/\mathfrak{m}}$ (with $k_1 > 0$, $k_2 > 0$, $\omega_2 > \omega_1$). For many waves with overlap parameters $s \gg 1$, the relevant phase velocity range for waves influencing

the particle is a "resonance box", with a width scaling as $(A/\mathfrak{m})^{2/3}$. [BeEs97, BeEs98a]

A good approximation to typical wave dispersion relations in the strong overlap limit, after a Galileo change of reference frame (see e.g. sec. 6.7 in [ElEs03]), is

$$(2.3) \qquad k_m = k_0, \ \omega_m = 2\pi(m - \mathcal{M}/2)/\mathcal{T},$$

for some k_0, $\mathcal{T}$. Then, in the limit $\mathcal{M} \to \infty$, the equations of motion yield for $A_m \mathrm{e}^{\mathrm{i}\varphi_m} = A_0$ (with real A_0) the well-known standard map [BeEs98b]. The case where phases φ_m are independent random variables uniformly distributed on the circle $[0, 2\pi]$, while $A_m = A_0$ is given, was investigated notably by Cary, Escande, Verga and Bénisti [BeEs97, CEV90, ElEs03] and occurs in the context of the random phase approximation.

To the extent that the phases and amplitudes of the waves are independent random variables, the physicist usually views the force (2.2) as a mollification of a white noise, with amplitude $\sigma = \sqrt{\mathbb{E}(k_m^2 A_m^2)} = |k_m|\sqrt{\mathbb{E}A_m^2}$, where the relevant mode m is the one nearest to the current particle velocity (the mathematical expectation $\mathbb{E}$ is called ensemble average with respect to wave amplitudes and phases).[1]

This is the core of quasilinear theory [DrPi62, Pe94, RoFi61, VVS62]. With some care, one interprets (2.1)–(2.2) as a stochastic differential equation ; this applies in the case $m \in \mathbb{Z}$ for dispersion relation (2.3) with Gaussian independent complex amplitudes such that $\mathbb{E}A_m^2 = k_0^{-2}\sigma^2$. The particle velocity then has a Brownian evolution, so that for $t, t' \in [0, \mathcal{T}]$

$$(2.4) \qquad \mathbb{E}(p_t - p_0)(p_{t'} - p_0) = D_{\mathrm{QL}} \min(t, t')$$

with the quasilinear diffusion coefficient $D_{\mathrm{QL}} = \sigma^2 \mathcal{T}/2$. However, the particle evolution for $t > \mathcal{T}$ may show a strong correlation to its motion for $0 \le t \le \mathcal{T}$ because the waves are periodic in time [El07, El08], and the dispersion relation (2.3) may generate a strong spatial correlation between the motions of two particles because all waves acting on a particle at any time have the same wavelength.[2]

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[1]Phases do not appear in σ (nor in s) because the relative phase of two waves $\varphi_m + \omega_m t - \varphi_n - \omega_n t$ varies uniformly over time (hence $\varphi_m - \varphi_n$ can be absorbed in the choice of the time origin).

[2] There is a large body of literature on the case of incoherent waves with no dispersion relation. Then the sum $\sum_m$ becomes a double sum $\sum_{m,n}$ and one varies wavenumbers k_n independently from pulsations ω_m. This space-time stochastic environment is more noisy than our model and may also be considered to motivate a quasilinear approximation.

Set $k_0 = 1$ and $\mathcal{T} = 2\pi$ by the choice of space and time units. The particle phase space is $\mathbb{T} \times \mathbb{R}$, where $\mathbb{T}$ is the circle modulo 2π, and the particle equation of motion reads, with initial data $(q(0), \dot{q}(0)) = (q_0, \dot{q}_0)$,

$$
(2.5) \quad
\begin{aligned}
\ddot{q} &= \mathfrak{m}^{-1} \sum_{m=-\mu}^{\mu'-1} A_m \sin(q - mt + \varphi_m) \\
&= \mathfrak{m}^{-1} \sum_m A_m \cos(mt - \varphi_m) \sin q - \mathfrak{m}^{-1} \sum_m A_m \sin(mt - \varphi_m) \cos q
\end{aligned}
$$

in a galilean frame moving at a velocity inside the spectrum of wave phase velocities ($\mu + \mu' = \mathcal{M}$). We assume $\mu \gg 1$, $\mu' \gg 1$. For $A_m = A_0$ given and independent random phases (uniform on the circle), in the limit $A_0/\mathfrak{m} \to \infty$, with $(\mathfrak{m}/A_0)^{2/3} \mathcal{M} \gtrsim 10$, Bénisti and Escande [BeEs97, BeEs98a] have shown that the particle momentum $p = \mathfrak{m}\dot{q}$ follows essentially a Brownian motion, with diffusion constant given by the quasilinear estimate

$$
D_{\mathrm{QL}} = \pi A_0^2
$$

as long as the motion does not approach the boundaries of the wave velocity spectrum. Then the particle momenta p for an ensemble of independent realizations of the system will be described by a distribution function verifying the Fokker–Planck equation

$$
(2.6) \quad \partial_t f = \frac{D_{\mathrm{QL}}}{2} \partial_p^2 f
$$

even for $t > \mathcal{T}$. Numerical simulations [El08] show similar behaviour for i.i.d. complex Gaussian random variables $A_m \mathrm{e}^{\mathrm{i}\varphi_m}$. The present work establishes a rigorous version of this result in the frame of stochastic processes.

3. Main results. We first let $\min(\mu, \mu') \to \infty$ in the model, taking $A_m \cos \varphi_m$, $A_m \sin \varphi_m$ as i.i.d. Gaussian random variables with zero expectation and $\mathbb{E}(A_m \cos \varphi_m)^2 = \mathbb{E}(A_m \sin \varphi_m)^2 = 1/2$ for all $m \in \mathbb{Z}$. In particular this implies that phases φ_m are i.i.d. uniformly on $\mathbb{T}$. To follow earlier practice, we now set $\mathfrak{m} = 1/A$. Then formally (2.5) becomes the Stratonovich stochastic differential equation for $0 \le t \le 2\pi$

$$
\begin{aligned}
(3.1) \qquad \mathrm{d}Q_t &= A P_t \, \mathrm{d}t, \\
(3.2) \qquad \mathrm{d}P_t &= \sin(Q_t) \circ \mathrm{d}C_t + \cos(Q_t) \circ \mathrm{d}S_t,
\end{aligned}
$$

with initial data $Q_0 = q_0$, $P_0 = p_0 = \mathfrak{m}\dot{q}_0 = \dot{q}_0/A$; here, from [Ka85], $\pi^{-1/2}(C, S)$ is a standard 2-dimensional Brownian motion. In other words, C and S are martingales, with $C_0 = S_0 = 0$ and

$$
(3.3) \qquad \langle C \rangle_t = \langle S \rangle_t = \pi t, \quad \langle C, S \rangle_t = 0.
$$

Note that the Stratonovich and Itô integrals define the same solutions for this system, and that the vector fields $(\sin q)\partial_p$ and $(\cos q)\partial_p$ commute.

For $0 \le t \le 2\pi$, it is clear that P is a Brownian motion for any value of $A \ge 0$ (see the first lines of Sec. 7). However, the model (2.5) defines a dynamical system for all times $0 \le t < \infty$, and one may wonder how the initial stochastic behaviour over $[0, 2\pi]$ extends for longer times. Formally, one solves then (3.1)–(3.2) with the periodized field, i.e. with the continuous processes defined by

$$\text{(3.4)} \qquad\qquad \mathrm{d}C_{t+2k\pi} = \mathrm{d}C_t \quad , \quad \mathrm{d}S_{t+2k\pi} = \mathrm{d}S_t$$

for $k \in \mathbb{Z}$. In other words, $C_t - \frac{t}{2\pi}C_{2\pi}$ and $S_t - \frac{t}{2\pi}S_{2\pi}$ are independent Brownian bridges repeated periodically for $t \in \mathbb{R}$, while $C_{2\pi}$ and $S_{2\pi}$ are independent Gaussian random variables with expectation 0 and variance $2\pi^2$.

For this extended process, the wave field acting on the particle for $t \notin [0, 2\pi]$ is not stochastically independent from the wave field acting during $[0, 2\pi]$. Therefore one does not expect the particle momentum to proceed as a Brownian motion for all times, and indeed for A small enough the velocity $\dot{q}$ may remain bounded in a narrow interval for all times. This is easily seen numerically and can be attributed to the existence of Kolmogorov-Arnol'd-Moser invariant tori in the 3-dimensional extended phase space with coordinates $(t, q, \dot{q})$.

On the other hand, for large A, the dynamics viewpoint [BeEs97, BeEs98a, ElEs03] suggests that the nonlinearity in the equations of motion (due to trigonometric functions of Q) may enable a decorrelation of the force over the period $\mathcal{T} = 2\pi$, so that the long-time evolution of the velocity would also be close to Brownian. This is what we shall show.

An intimately related issue is the relative motion of several particles, released in the same realization of the wave field. Even though each particle velocity diffuses for $t \in [0, 2\pi]$, their motions are not independent. We shall also show that for large A the motions of any finite family of particles released at initial data $(Q_0^{(\nu)}, P_0^{(\nu)})$, $1 \le \nu \le \mathcal{N}$, approaches a family of $\mathcal{N}$ independent processes. This can also be expected from the consideration of the top Lyapunov exponent of the dynamics (3.1)–(3.2) in the limit $A \to \infty$.

THEOREM 3.1. *For any $\mathcal{N} > 0$, the momentum processes $P^{(\nu)}$ defined by*

$$\text{(3.5)} \qquad \mathrm{d}Q_t^{(\nu)} \;=\; AP_t^{(\nu)}\,\mathrm{d}t\,, \quad Q_0^{(\nu)} = q_0^{(\nu)}\,,$$

$$\text{(3.6)} \qquad \mathrm{d}P_t^{(\nu)} \;=\; (\sin Q_t^{(\nu)})\mathrm{d}C_t + (\cos Q_t^{(\nu)})\mathrm{d}S_t\,, \quad P_0^{(\nu)} = p_0^{(\nu)}\,,$$

with $\mathcal{N}$ different initial data $(q_0^{(\nu)}, p_0^{(\nu)}) \in \mathbb{T} \times \mathbb{R}$, $1 \leq \nu \leq \mathcal{N}$, converge as $A \to \infty$ to $\mathcal{N}$ independent Wiener processes with variance πt, and convergence is in law in $C(\mathbb{R}_+, \mathbb{R}^{\mathcal{N}})$.

The key argument in the proof is the following weak convergence theorem, where we write now $n = \pi^{1/2} A$. Consider the two–dimensional diffusion process indexed by $n \geq 1$, solution of the SDE on $\mathbb{R}^2$

(3.7)
$$
\begin{cases}
\dfrac{\mathrm{d}U_t^n}{\mathrm{d}t} = nV_t^n\,, \ U_0 = u\,, \\[2mm]
\mathrm{d}V_t^n = \sin(U_t^n)\mathrm{d}W_t\,, \ V_0 = v\,,
\end{cases}
$$

where $(u, v) \notin \{(k\pi, 0), \ k \in \mathbb{Z}\}$ and W is a standard Brownian motion. We prove

THEOREM 3.2. *As $n \to \infty$,*

$$
V^n \Rightarrow v + \frac{1}{\sqrt{2}}B\,,
$$

where $\{B_t, \ t \geq 0\}$ is a standard one–dimensional Brownian motion, and the convergence is in law in $C(\mathbb{R}_+, \mathbb{R})$.

4. A change of time scale. Note that for any $n \geq 1$, the law of $\{(U_t^n, V_t^n), \ t \geq 0\}$, the solution of (3.7), is characterized by the statement

$$
\begin{cases}
\dfrac{\mathrm{d}U_t^n}{\mathrm{d}t} = nV_t^n\,, \ U_0 = u\,, \\[2mm]
V^n \text{ is a martingale, } \dfrac{\mathrm{d}\langle V^n \rangle_t}{\mathrm{d}t} = \sin^2(U_t^n)\,, \ V_0^n = v\,.
\end{cases}
$$

Now define (like Bénisti and Escande [BeEs97, BeEs98a])

(4.1)
$$
X_t = U_{n^{-2/3}t}^n\,, \quad Y_t = n^{1/3}V_{n^{-2/3}t}^n\,.
$$

We first note that $X_0 = u$, $Y_0 = n^{1/3}v$, Y is a martingale, and

$$
\begin{cases}
\dfrac{\mathrm{d}X_t}{\mathrm{d}t} = n^{-2/3}\dfrac{\mathrm{d}U^n}{\mathrm{d}t}(n^{-2/3}t) = n^{1/3}V_{n^{-2/3}t}^n = Y_t\,, \\[2mm]
\langle Y \rangle_t = n^{2/3}\langle V^n \rangle_{n^{-2/3}t}\,, \ \dfrac{\mathrm{d}\langle Y \rangle_t}{\mathrm{d}t} = \sin^2(X_t)\,.
\end{cases}
$$

Using a well–known martingale representation theorem, we can pretend that there exists a standard Brownian motion $\{B_t, \ t \geq 0\}$ such that

(4.2)
$$
\begin{cases}
\dfrac{\mathrm{d}X_t}{\mathrm{d}t} = Y_t\,, \ X_0 = u\,, \\[2mm]
\mathrm{d}Y_t = \sin(X_t)\mathrm{d}B_t\,, \ Y_0 = n^{1/3}v\,.
\end{cases}
$$

Note that the process $\{(X_t, Y_t),\ t \geq 0\}$ still depends upon n, but only through the value of Y_0.

On the other hand, $V_t^n = n^{-1/3} Y_{n^{2/3}t}$. Hence

$$V_t^n = v + n^{-1/3} \int_0^{n^{2/3}t} \sin(X_s) \mathrm{d}B_s \,,$$

in other words V^n is a martingale such that $V_0^n = v$ and

$$\langle V^n \rangle_t = n^{-2/3} \int_0^{n^{2/3}t} \sin^2(X_s^n) \mathrm{d}s \,.$$

Here we recall the fact that the process X depends upon n (through the initial condition of Y), unless $v = 0$. Consequently

(4.3) $$\lim_{n \to \infty} \langle V^n \rangle_t = t \times \lim_{n \to \infty} \frac{1}{n^{2/3}t} \int_0^{n^{2/3}t} \sin^2(X_s^n) \mathrm{d}s$$

and in order to prove Theorem 3.2 it suffices to show that the above limit is $t/2$.

5. Qualitative properties of the solution of (4.2). We now consider the two–dimensional diffusion process

(5.1) $$\begin{cases} \dfrac{\mathrm{d}X_t}{\mathrm{d}t} = Y_t \,,\ X_0 = x \,, \\ \mathrm{d}Y_t = \sin(X_t)\mathrm{d}B_t \,,\ Y_0 = y \,, \end{cases}$$

with values in the state–space $E = [0, 2\pi) \times \mathbb{R} \setminus \{(0,0), (\pi, 0)\}$, where 2π is identified with 0. We first prove that the process $\{(X_t, Y_t),\ t \geq 0\}$ is a conservative E–valued diffusion. Indeed,

PROPOSITION 5.1. *Whenever the initial condition (x,y) belongs to E,*

$$\inf\{t > 0,\ (X_t, Y_t) \in \{(0,0), (\pi, 0)\}\} = +\infty \quad a.s.$$

PROOF: We define the stopping time

$$\tau = \inf\{t,\ (X_t, Y_t) = (0,0)\} \,.$$

Let $R_t = X_t^2 + Y_t^2$, $Z_t = \log R_t$, $t \geq 0$. A priori, Z_t takes its values in $[-\infty, +\infty)$. Itô calculus on the interval $[0, \tau)$ yields

$$\begin{aligned} \mathrm{d}X_t^2 &= 2X_t Y_t \mathrm{d}t \,, \\ \mathrm{d}Y_t^2 &= 2Y_t \sin(X_t)\mathrm{d}B_t + \sin^2(X_t)\mathrm{d}t \,, \\ \mathrm{d}Z_t &= \frac{\mathrm{d}R_t}{R_t} - \frac{\mathrm{d}\langle R \rangle_t}{2R_t^2} \\ &= \frac{2Y_t X_t + \sin^2(X_t)}{R_t}\mathrm{d}t - 2\frac{Y_t^2 \sin^2(X_t)}{R_t^2}\mathrm{d}t + 2\frac{Y_t \sin(X_t)}{R_t}\mathrm{d}B_t \,. \end{aligned}$$

Now clearly $|\sin(x)| \le |x|$, $\sin^2(x) \le x^2$, and it follows from the above and standard inequalities that on the time interval $[0, \tau)$,

$$Z_t \ge Z_0 - 2t + \int_0^t \varphi_s \mathrm{d}B_s\,,$$

where $|\varphi_s| \le 1$. Hence the process $\{Z_t,\ t \ge 0\}$ is bounded from below on any finite time interval, which implies that $\tau = +\infty$ a.s., since $\tau = \inf\{t, Z_t = -\infty\}$. A similar argument shows that $\tau' = +\infty$ a.s., where

$$\tau' = \inf\{t,\ (X_t, Y_t) \in \{(0,0), (\pi, 0)\}\}\,.$$

$\square$

We next prove (here and below $\mathcal{B}_E$ stands for the σ–algebra of Borel subsets of E) the

PROPOSITION 5.2.   *The transition probabilities*

$$\{p((x,y); t, A) := \mathbb{P}_{x,y}((X_t, Y_t) \in A),\ (x, y) \in E,\ t > 0,\ A \in \mathcal{B}_E\}$$

have smooth densities $p((x, y); t, (x', y'))$ *with respect to Lebesgue's measure* $\mathrm{d}x'\mathrm{d}y'$ *on* E.

PROOF: Consider the Lie algebra of vector fields on E generated by $X_1 = \sin(x)\frac{\partial}{\partial y}$, $X_2 = [X_0, X_1]$ and $X_3 = [X_0, [X_0, X_1]]$, where $X_0 = y\frac{\partial}{\partial x}$. This Lie algebra has rank 2 at each point of E. The result is now a standard consequence of the well–known Malliavin calculus, see e.g. Nualart [Nu06]. $\square$

PROPOSITION 5.3.   *The* E–*valued diffusion process* $\{(X_t, Y_t),\ t \ge 0\}$ *is topologically irreducible, in the sense that for all* $(x, y) \in E$, $t > 0$, $A \in \mathcal{B}_E$ *with non empty interior,*

$$\mathbb{P}_{x,y}((X_t, Y_t) \in A) > 0\,.$$

PROOF: From Stroock–Varadhan's support theorem, see e.g. Ikeda–Watanabe [IkWa89], the support of the law of (X_t, Y_t) starting from $(X_0, Y_0) = (x, y)$ is the closure of the set of points which the following controlled ordinary differential equation can reach at time t by varying the control $\{u(s),\ 0 \le s \le t\}$

in the class of piecewise continuous functions :

(5.2)
$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}s}(s) = y(s)\,, & x(0) = x\,; \\[2mm] \dfrac{\mathrm{d}y}{\mathrm{d}s}(s) = \sin(x(s))u(s)\,, & y(0) = y\,. \end{cases}$$

It is not hard to show that the set of accessible points at time $t > 0$ by the solution of (5.2) is dense in E. The result now follows from the fact that the transition probability is absolutely continuous with respect to Lebesgue's measure, see Proposition 5.2. $\qquad\square$

We next prove the

LEMMA 5.4.
$$\mathbb{P}\left(|Y_t| \to \infty,\ \text{as } t \to \infty\right) = 0\,.$$

PROOF: The Lemma follows readily from the fact that

$$Y_t = W\left(\int_0^t \sin^2(X_s)\mathrm{d}s\right),$$

where $\{W(t),\ t \geq 0\}$ is a scalar Brownian motion. Then either $\int_0^t \sin^2(X_s)\mathrm{d}s$ is bounded and Y_t is finite, or the integral diverges and Y_t is finite anyway because W is recurrent. $\qquad\square$

Hence the topologically irreducible E–valued Feller process $\{(X_t, Y_t),\ t \geq 0\}$ is recurrent. Its unique (up to a multiplicative constant) invariant measure is the Lebesgue measure on E, so that in particular the process is null–recurrent. It then follows from (ii) in Theorem 20.21 from Kallenberg [Ka02]

LEMMA 5.5.   *For all $M > 0$, as $t \to \infty$,*

$$\frac{1}{t}\int_0^t \mathbf{1}_{\{|Y_s|\leq M\}}\mathrm{d}s \to 0 \quad a.s.$$

6. A path decomposition of the process $\{(X_t, Y_t),\ t \geq 0\}$. We first define two sequences of stopping times. Let $T_0 = 0$ and

$$\text{for } \ell \text{ odd,} \quad T_\ell = \inf\{t > T_{\ell-1},\ |Y_t| \geq M + 1\}\,,$$
$$\text{for } \ell \text{ even,} \quad T_\ell = \inf\{t > T_{\ell-1},\ |Y_t| \leq M\}\,.$$

Let now $\tau_0 = T_1$. We next define recursively $\{\tau_k, \ k \geq 1\}$ as follows. Given τ_{k-1}, we first define

$$L_k = \sup\{\ell \geq 0, \ \tau_{k-1} \geq T_{2\ell+1}\}.$$

Now let

$$\eta_k = \begin{cases} \tau_{k-1} & \text{if } \tau_{k-1} < T_{2L_k+2}, \\ T_{2L_k+3} & \text{if } \tau_{k-1} \geq T_{2L_k+2}. \end{cases}$$

We now define

$$\tau_k = \inf\{t > \eta_k, \ |X_t - X_{\eta_k}| = 2\pi\} \wedge \inf\{t > \eta_k, \ |Y_t - Y_{\eta_k}| > 1\}.$$

It follows from the above definitions that

$$\int_0^t \mathbf{1}_{\{|Y_s| \geq M+1\}} \sin^2(X_s)\mathrm{d}s \leq \sum_{k=1}^\infty \int_{\eta_k \wedge t}^{\tau_k \wedge t} \sin^2(X_s)\mathrm{d}s \leq \int_0^t \sin^2(X_s)\mathrm{d}s,$$

a statement which will be refined in the proof of Proposition 6.3. Define

$$\begin{aligned}
K^0 &= \{k \geq 1, \ |Y_{\tau_k} - Y_{\eta_k}| < 1\}, \\
K^1 &= \{k \geq 1, \ |Y_{\tau_k} - Y_{\eta_k}| = 1\}, \\
K_t &= \{k \geq 1, \ \eta_k < t\}, \\
K_t^0 &= K^0 \cap K_t, \\
K_t^1 &= K^1 \cap K_t.
\end{aligned}$$

We first prove the

LEMMA 6.1.
$$\frac{1}{t} \sum_{k \in K_t^1} (\tau_k - \eta_k) \to 0$$

in $L^1(\Omega)$ as $M \to \infty$, uniformly in $t > 0$.

PROOF: We shall use repeatedly the fact that since $|Y_{\eta_k}| \geq M > 2$, $|Y_{\eta_k}| - 1 \geq |Y_{\eta_k}|/2$. We have that (see the Appendix below), since $\tau_k - \eta_k \leq 4\pi/|Y_{\eta_k}|$,

$$\begin{aligned}
\mathbb{P}(k \in K^1 | \mathcal{F}_{\eta_k}) &\leq& \mathbb{P}\left(\sup_{\eta_k \leq t \leq \tau_k} |Y_t - Y_{\eta_k}| \geq 1 \Big| \mathcal{F}_{\eta_k}\right) \\
&\leq& 2\exp(-|Y_{\eta_k}|/(8\pi)).
\end{aligned}$$

Consequently, using again the inequality $\tau_k - \eta_k \leq 4\pi/|Y_{\eta_k}|$, we deduce that

$$
\begin{aligned}
\mathbb{E}\left[(\tau_k - \eta_k)\mathbf{1}_{\{k \in K^1\}}|\mathcal{F}_{\eta_k}\right] &\leq \frac{8\pi}{|Y_{\eta_k}|}\exp(-|Y_{\eta_k}|/(8\pi)) \\
&\leq \frac{8\pi}{|Y_{\eta_k}|}\exp(-M/(8\pi)) \,.
\end{aligned}
$$

On the other hand, whenever $k \in K^0$,

$$
\tau_k - \eta_k \geq 2\pi/(|Y_{\eta_k}| + 1) \geq \pi/|Y_{\eta_k}| \,.
$$

Now, provided $t \geq 4\pi/M$,

$$
\begin{aligned}
2t &\geq t + \frac{4\pi}{M} \\
&\geq \mathbb{E}\left[\sum_{k \in K_t^0}(\tau_k - \eta_k)\right] \\
&\geq \pi\mathbb{E}\left[\sum_{k \in K_t}\mathbf{1}_{\{k \in K^0\}}\frac{1}{|Y_{\eta_k}|}\right] \\
&\geq \frac{\pi}{2}\mathbb{E}\left[\sum_{k \in K_t}\frac{1}{|Y_{\eta_k}|}\right] \,,
\end{aligned}
$$

since

$$
\begin{aligned}
\mathbb{P}(k \in K^0|\mathcal{F}_{\eta_k}) &= 1 - \mathbb{P}(k \in K^1|\mathcal{F}_{\eta_k}) \\
&\geq 1 - 2\exp(-M/(8\pi)) \\
&\geq 1/2 \,,
\end{aligned}
$$

provided M is large enough. Finally

$$
\begin{aligned}
\frac{1}{t}\mathbb{E}\left[\sum_{k \in K_t^1}(\tau_k - \eta_k)\right] &\leq 32\exp(-M/(8\pi))\frac{\mathbb{E}\left[\sum_{k \in K_t}|Y_{\eta_k}|^{-1}\right]}{\mathbb{E}\left[\sum_{k \in K_t}|Y_{\eta_k}|^{-1}\right]} \\
&= 32\exp(-M/(8\pi)) \\
&\to 0 \,,
\end{aligned}
$$

as $M \to \infty$, uniformly in t. $\qquad\square$

Now, for any $k \in K^0$,

$$
\begin{aligned}
\int_{\eta_k}^{\tau_k} \sin^2(X_s)\mathrm{d}s \;=\;& \frac{\tau_k - \eta_k}{2\pi} \int_0^{2\pi} \sin^2(x)\mathrm{d}x \\
& + \int_{\eta_k}^{\tau_k} \sin^2(X_s)\left[1 - \frac{Y_s(\tau_k - \eta_k)}{2\pi}\right]\mathrm{d}s\,,
\end{aligned}
$$

(6.1)

and we have

$$
\begin{aligned}
\left|\int_{\eta_k}^{\tau_k} \sin^2(X_s)\left[1 - \frac{Y_s(\tau_k - \eta_k)}{2\pi}\right]\mathrm{d}s\right| &= \left|\int_{\eta_k}^{\tau_k}\int_{\eta_k}^{\tau_k} \sin^2(X_s)\frac{Y_r - Y_s}{2\pi}drds\right| \\
&\le \frac{1}{2\pi}\int_{\eta_k}^{\tau_k}\int_{\eta_k}^{\tau_k} |Y_r - Y_s|drds\,.
\end{aligned}
$$

Finally we have the

LEMMA 6.2. *Uniformly in $t > 0$,*

$$
\frac{\sum_{k\in K_t^0} \int_{\eta_k}^{\tau_k}\int_{\eta_k}^{\tau_k} |Y_r - Y_s|drds}{\sum_{k\in K_t^0}(\tau_k - \eta_k)} \to 0
$$

a.s., as $M \to \infty$.

PROOF: Since $|Y_t - Y_{\eta_k}| \le 1$ for $\eta_k \le t \le \tau_k$,

$$
\begin{aligned}
\frac{\sum_{k\in K_t^0} \int_{\eta_k}^{\tau_k}\int_{\eta_k}^{\tau_k} |Y_r - Y_s|drds}{\sum_{k\in K_t^0}(\tau_k - \eta_k)} &\le 2 \sup_{k\in K_t^0}(\tau_k - \eta_k) \\
&\le 8\pi/M \\
&\to 0\,,
\end{aligned}
$$

as $M \to \infty$, uniformly in t. □

We are now in a position to prove the following ergodic type theorem, from which Theorem 3.2 will follow :

PROPOSITION 6.3. *As $t \to \infty$,*

$$
\frac{1}{t}\int_0^t \sin^2(X_s)\mathrm{d}s \to \frac{1}{2}
$$

in probability.

PROOF: We first note that

$$[0,t] = B_t^0 \cup B_t^1 \cup C_t \,,$$

where

$$B_t^0 = [0,t] \cap \left(\cup_{k \in K_t^0} [\eta_k, \tau_k] \right) \,,$$
$$B_t^1 = [0,t] \cap \left(\cup_{k \in K_t^1} [\eta_k, \tau_k] \right) \,,$$
$$C_t = [0,t] \backslash (B_t^0 \cup B_t^1) \,.$$

We have

$$\frac{1}{t} \int_0^t \sin^2(X_s) \mathrm{d}s = \frac{1}{t} \int_0^t \mathbf{1}_{B_t^0}(s) \sin^2(X_s) \mathrm{d}s + \frac{1}{t} \int_0^t \mathbf{1}_{B_t^1}(s) \sin^2(X_s) \mathrm{d}s$$
$$+ \frac{1}{t} \int_0^t \mathbf{1}_{C_t}(s) \sin^2(X_s) \mathrm{d}s \,.$$

Now $C_t \subset \{s \in [0,t], \; |Y_s| \le M+1\}$, so for each fixed $M > 0$, it follows from Lemma 5.5 that the last term can be made arbitrarily small, by choosing t large enough. The second term goes to zero as $M \to \infty$, uniformly in t, from Lemma 6.1. Finally the first term equals the searched limit, plus an error term which goes to 0 as $M \to \infty$, uniformly in t, see Lemma 6.2 and the following fact, which follows from the combination of Lemma 6.1 and Lemma 5.5 :

$$\frac{1}{t} \sum_{k \in K_t^0} (\tau_k - \eta_k) \to 1$$

in probability, as $n \to \infty$.   □

We can finally proceed with the

PROOF OF THEOREM 3.2 All we have to show is that (see (4.3))

$$\lim_{n \to \infty} \frac{1}{n^{2/3}t} \int_0^{n^{2/3}t} \sin^2(X_s^n) \mathrm{d}s = \frac{1}{2\pi} \int_0^{2\pi} \sin^2(x') \mathrm{d}x' = \frac{1}{2}$$

in probability. In the case $v = 0$, the process $\{(X_t^n, Y_t^n)\}$ does not depend upon n, and the result follows precisely from Proposition 6.3. Now suppose that $v \ne 0$. In that case, the result can be reformulated equivalently as follows. For some $x \in \mathbb{R}$, $y \ne 0$, each $t > 0$, define the process $\{(X_s^t, Y_s^t), \; 0 \le s \le t\}$ as the solution of the SDE

$$\begin{cases} \dfrac{\mathrm{d}X_s^t}{\mathrm{d}s} = Y_s^t \,, \; X_0^t = x \,, \\ \mathrm{d}Y_s^t = \sin(X_s^t) \, \mathrm{d}W_s \,, \; Y_0^t = \sqrt{t}\, y \,. \end{cases}$$

We need to show that

$$\frac{1}{t}\int_0^t \sin^2(X_s^t)\mathrm{d}s \to \frac{1}{2\pi}\int_0^{2\pi} \sin^2(x')\mathrm{d}x'$$

in probability, as $t \to \infty$. Note that in time t, the process Y^t starting from $\sqrt{t}y$ can come back near the origin.

It is easily seen, by introducing the Markov time $\tau_M^t = \inf\{s > 0, \ |Y_s^t| \le M\}$ and exploiting the strong Markov property, that

$$\frac{1}{t}\int_0^t \mathbf{1}_{\{|Y_s^t|\le M\}}\mathrm{d}s \to 0 \quad \text{a.s.}$$

follows readily from Lemma 5.5. The rest of the argument leading to Proposition 6.3 is based upon limits as $M \to \infty$, uniformly with respect to t. It thus remains to check that the fact that Y_0^t now depends upon t does not spoil this uniformity, which is rather obvious. $\qquad\square$

REMARK 6.4. This proof holds uniformly with respect to initial data (u, v) satisfying $|v| \ge a$ for any $a > 0$.

7. Proof of Theorem 3.1. Our Theorem 3.1 now appears as a simple corollary of Theorem 3.2.

PROOF: We first prove the Theorem for $t \in [0, 2\pi]$. Then the vector $\mathbf{P} = (P^{(1)}, \dots, P^{(\mathcal{N})})$ is a martingale in $\mathbb{R}^{\mathcal{N}}$, and to prove our claim it suffices to show that its quadratic variation matrix converges to πt times the identity matrix. The diagonal elements of the matrix are

$$\langle P^{(\nu)}\rangle_t = \int_0^t (\sin^2 Q_s^{(\nu)} + \cos^2 Q_s^{(\nu)})\pi\mathrm{d}s = \pi t$$

and we only need to compute the cross-variation

$$\langle P^{(\nu)}, P^{(\nu')}\rangle_t = \int_0^t (\sin Q_s^{(\nu)} \sin Q_s^{(\nu')} + \cos Q_s^{(\nu)} \cos Q_s^{(\nu')})\pi\mathrm{d}s$$

$$(7.1) \qquad\qquad = \int_0^t \cos(Q_s^{(\nu)} - Q_s^{(\nu')})\pi\mathrm{d}s.$$

Now, define (with $n = \pi^{1/2}A$)

$$U_t^n = \frac{1}{2}(Q_t^{(\nu)} - Q_t^{(\nu')}) \quad , \quad V_t^n = n^{-1}\frac{\mathrm{d}U_t^n}{\mathrm{d}t},$$

$$U_t'^n = \frac{1}{2}(Q_t^{(\nu)} + Q_t^{(\nu')}) \quad , \quad V_t'^n = n^{-1}\frac{\mathrm{d}U_t'^n}{\mathrm{d}t}.$$

These processes solve the stochastic differential equation

$$(7.2) \qquad \mathrm{d}U_t^n = nV_t^n \, \mathrm{d}t \,, \quad U_0^n = \frac{q_0^{(\nu)} - q_0^{(\nu')}}{2} \,,$$

$$(7.3) \qquad V_0^n = \frac{p_0^{(\nu)} - p_0^{(\nu')}}{2\pi^{1/2}} \,,$$

$$\mathrm{d}V_t^n = \frac{1}{2\sqrt{\pi}} \left(\sin(U_t'^n + U_t^n) - \sin(U_t'^n - U_t^n) \right) \mathrm{d}C_t$$

$$+ \frac{1}{2\sqrt{\pi}} \left(\cos(U_t'^n + U_t^n) - \cos(U_t'^n - U_t^n) \right) \mathrm{d}S_t$$

$$(7.4) \qquad = \sin U_t^n \, \mathrm{d}W_t^n$$

where the process W^n is the martingale defined by $W_0^n = 0$ and

$$\mathrm{d}W_t^n = \pi^{-1/2}(\cos U_t'^n \, \mathrm{d}C_t - \sin U_t'^n \, \mathrm{d}S_t) \,.$$

The quadratic variation of W^n is

$$\langle W^n \rangle_t = \pi^{-1} \int_0^t (\cos^2 U_s'^n + \sin^2 U_s'^n) \, \pi \, \mathrm{d}s = t$$

in view of the quadratic and cross variations (3.3) of (C, S), and this result does not depend on the process U'^n (which follows the center of mass of the two particles ν and ν'). Thus W^n is a standard Wiener process, and (U^n, V^n) defined by (7.2)–(7.3)–(7.4) satisfies the hypotheses of Theorem 3.2. Hence, with X defined by (4.1),

$$\langle P^{(\nu)}, P^{(\nu')} \rangle_t = \int_0^t \cos(2U_s^n)\pi \mathrm{d}s = \int_0^t (1 - 2\sin^2 U_s^n)\pi \mathrm{d}s$$

$$= \pi t \left(1 - \frac{2}{n^{2/3}t} \int_0^{n^{2/3}t} \sin^2 X_{s'}^n \mathrm{d}s' \right)$$

which converges in probability to 0 for $n \to \infty$ as shown in the proof of the theorem.

Now we consider the process over the interval $[0, 4\pi]$, taking into account that (C, S) over the whole interval is neither a martingale nor Markov. From the given initial data, amplitude A and realization of (C, S), we define a subsidiary set of particles $\mathcal{N} + 1 \leq \nu \leq 2\mathcal{N}$, with initial data

$$Q_0^{(\nu)} = Q_{2\pi}^{(\nu-\mathcal{N})} \,, \ P_0^{(\nu)} = P_{2\pi}^{(\nu-\mathcal{N})} \,,$$

of which a.s. none coincides (modulo 2π for q) with any of the initial data $(Q_0^{(\nu)}, P_0^{(\nu)})$. Recalling that at $t = 2\pi$ the law of any $P_{2\pi}^{(\nu)}$ (for $1 \leq \nu \leq \mathcal{N}$)

is Gaussian with variance $2\pi^2$ (so that its probability density is bounded by $1/\sqrt{2\pi\ 2\pi^2} < 1/(2\pi)$), this ensures that, given $\epsilon > 0$,

$$\mathbb{P}\left(\min_{1\leq\nu,\nu'\leq\mathcal{N}}|P_0^{(\nu)} - P_{2\pi}^{(\nu')}| < \epsilon \text{ or } \min_{1\leq\nu<\nu'\leq\mathcal{N}}|P_{2\pi}^{(\nu)} - P_{2\pi}^{(\nu')}| < \epsilon\right)$$

$$\leq \quad \frac{3\epsilon\mathcal{N}^2}{\sqrt{2\pi\ 2\pi^2}} < \epsilon\mathcal{N}^2/2\,.$$

For processes such that $\min_{\nu,\nu'}|P_0^{(\nu)} - P_{2\pi}^{(\nu')}| \geq \epsilon$ and $\min_{\nu<\nu'}|P_{2\pi}^{(\nu)} - P_{2\pi}^{(\nu')}| \geq \epsilon$, the above proof holds (uniformly with respect to initial data) for the full set of $2\mathcal{N}$ particles defined here. It follows that a.s. the $P_t^{(\nu)} - P_0^{(\nu)}$, $t \in [0, 2\pi]$, $1 \leq \nu \leq 2\mathcal{N}$ are mutually independent Wiener processes, each with variance πt. This implies that $P_t^{(\nu)} - P_0^{(\nu)}$, $t \in [0, 4\pi]$, $1 \leq \nu \leq \mathcal{N}$ are also mutually independent Wiener processes.

For any interval $[0, 2k\pi]$ with $k \in \mathbb{N}_0$ the same argument reduces the $\mathcal{N}$-particle problem to $k\mathcal{N}$ particles over $[0, 2\pi]$, and the proof is complete. $\square$

REMARK 7.1.   Our theorem allows that $P_0^{(\nu)} = P_0^{(\nu')}$ for some $\nu \neq \nu'$ with $1 \leq \nu, \nu' \leq \mathcal{N}$.

8. Perspectives.   The implications of Theorem 3.1 are twofold. First, for $\mathcal{N} = 1$, they support the observation [BeEs97, BeEs98a, CEV90, El08] that, in systems with finite $\mathcal{M} \gg 1$ and $A_0/\mathfrak{m} \gg 1$ (with appropriate scaling), the long-time behaviour of a single particle in a periodic wave field exhibits statistical properties approaching those of the Brownian motion. To complete the connection with physics literature, one must now discuss how the finite sums in (2.2) approach the right hand side of (3.2), and how this implies that the solutions of the first equation approach the solutions of the latter equation [Su78]. This will be discussed separately.

Second, and this is conceptually more fundamental, for a single sample of the Wiener wavefield (with $\mathcal{M} = \infty$ formally), with strong spatial correlations (thanks to the single wavevector k_0 in the model), the limit $A_0/\mathfrak{m} \to \infty$ leads to independence of the evolutions of all $\mathcal{N}$ particles, which can then be collectively described by the diffusion equation. While such an independence is often admitted without proof in physics practice, our work provides an explicit justification to it. We even prove a little more than usual one– or two–time statements, as in our limit $p^{(\mathcal{N})}$ is independent in law from even the full evolution data $\{p^{(\nu)}(t),\ 0 \leq t \leq T,\ 1 \leq \nu \leq \mathcal{N} - 1\}$.

This second implication is an important issue, as the acceleration of a passive particle is a Hamiltonian process while the diffusion equation is irreversible. While the key to this irreversibility is clearly the fact that the diffusion process only relates to the momentum component of particle evolution, we shall further investigate the interplay of limits $\mathcal{N} \to \infty$ and $A \to \infty$, or $\mathfrak{m} \to 0$, with stochasticity versus Hamiltonian "conservativeness" in (Q, P) variables in future work.

Finally, we followed general practice in discussing the stochastic acceleration problem in only one space dimension [Chi79, HaWa04]. This is rather classical, and it applies e.g. to particle motion along magnetic field lines in strongly magnetized plasmas ; higher-dimensional motions may call for different elementary models.

APPENDIX A

For the convenience of the reader, we prove the following well–known result (see e.g. ex. IV.3.16 in [ReYo99])

PROPOSITION A.1.   *Let η and τ be two stopping times such that $0 \leq \eta \leq \tau \leq \eta + T$ and $M_t = \int_0^t \varphi_s \mathrm{d}B_s$, where $\{B_t, \ t \geq 0\}$ is a standard Brownian motion and $\{\varphi_t, \ t \geq 0\}$ is progressively measurable and satisfies $\int_0^t \varphi_s^2 \mathrm{d}s \leq k^2 t$ for all $t \geq 0$. Then for all $b > 0$,*

$$\mathbb{P}\left(\sup_{\eta \leq t \leq \tau} |M_t - M_\eta| \geq b\right) \leq 2 \exp\left(-\frac{b^2}{2k^2 T}\right).$$

PROOF: From the optional stopping theorem, it suffices to treat the case $\eta = 0$, $\tau = T$. We have

$$\mathbb{P}(\sup_{0 \leq t \leq T} |M_t| \geq b) \leq \mathbb{P}(\sup_{0 \leq t \leq T} M_t \geq b) + \mathbb{P}(\inf_{0 \leq t \leq T} M_t \leq -b).$$

We estimate the first term on the right. The second one is bounded by the same quantity. Define for all $\lambda > 0$

$$\mathcal{M}_t^\lambda = \exp\left(\lambda M_t - \frac{\lambda^2}{2} \int_0^t \varphi_s^2 \mathrm{d}s\right).$$

Then

$$\mathbb{P}(\sup_{0 \leq t \leq T} M_t \geq b) \leq \mathbb{P}\left(\sup_{0 \leq t \leq T} \mathcal{M}_t^\lambda \geq \exp(\lambda b - \lambda^2 k^2 T/2)\right)$$
$$\leq \exp(\lambda^2 k^2 T/2 - \lambda b),$$

from Doob's inequality, since $\{\mathcal{M}_t^\lambda, \ t \geq 0\}$ is a martingale with mean one. Optimizing the value of λ, we deduce that

$$\mathbb{P}\left(\sup_{0 \leq t \leq T} M_t \geq b\right) \leq \exp\left(-\frac{b^2}{2k^2T}\right),$$

from which the result follows. $\qquad\square$

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